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TWIST MAPS, COVERINGS AND BROUWER'S TRANSLATION THEOREM

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ABSTRACT. We apply the Brouwer Translation Theorem to a class of twist maps of the annulus (which contains C^1 area preserving maps) to show that, if h belongs to this class, then a certain set $\mathcal{P}_0$ of periodic points of h cannot be dense. The definition of $\mathcal{P}_0$ does not impose any a priori restrictions on the periods of the points of $\mathcal{P}_0$.

Introduction. Let h: $R^2 \to R^2$ be an orientation-preserving, fixed point free, self-homeomorphism of the two-dimensional plane.

The weakest version of Brouwer's Translation Theorem (see e.g. [1])[2] states: if $A \subset R^2$ is an arc-wise connected set such that $A \cap h(A) = \varnothing$, then $A \cap h^n(A) = \varnothing$ for all $n \neq 0$. Here h^n denotes the n-fold iterate of h.

In order to see whether this strong result on the behavior of all iterates of h (i.e., a result of dynamics) has any relevance for the dynamics on other surfaces (especially compact surfaces, where all this is more interesting) it is natural to proceed as follows: Let h: $M \to M$ be an orientation-preserving self-homeomorphism of a connected, orientable two-manifold M (with or without boundary ∂M) such that its fixed point set, Fix h, does not separate M; let $M_0 = M - (\partial M \cup \text{Fix } h)$ and $h_0 = h|M_0$; then, since the universal cover $\tilde{M}_0$ of M_0 is homeomorphic to R^2 and any lifting to $\tilde{M}_0$, $\tilde{h}_0$: $\tilde{M}_0 \to \tilde{M}_0$, of h_0 is orientation-preserving and fixed point free, we can apply Brouwer's Translation Theorem to $\tilde{h}_0$ and ask whether, in this way, we obtain any relations downstairs on the dynamics of h_0: $M_0 \to M_0$, which is the same as that of the original homeomorphism h: $M \to M$.

We prove that the answer is yes, by using this idea to find a relation on a certain set $\mathcal{P}_0$ of periodic points of h, when h: $A \to A$ is a twist homeomorphism, which satisfies a mild purely homotopy-theoretical condition (see Definition 1.1, below). *The relation is that* $\mathcal{P}_0$ *cannot be dense* and so, in particular, if h is also topologically transitive, $\mathcal{P}_0$ is nowhere dense.

If h is at least C^1, then the same proof shows that $\bigcup H_0(P)$, the union taken over all hyperbolic $P \in \mathcal{P}_0$, is not dense either. Here $H_0(P)$ denotes a certain set of *homoclinic* points of P (see Definition 1.3, below).

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[2] Also S. A. Andrea, Trans. Amer. Math. Soc. **151** (1970), 481; Bull. Amer. Math. Soc. **71** (1965), 381.

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1. Statement of the Theorem. Let $A = \{(x, y) \in R^2 | 0 < a^2 \leq x^2 + y^2 \leq b^2\}$ denote a planar annulus and $h: A \to A$ an orientation-preserving homeomorphism which maps each component of the boundary of A onto itself. Let $\bar{A}$ denote the band $\bar{A} = \{(x, y) \in R^2 | a \leq y \leq b\}$, which we regard as the universal cover of A and $p: \bar{A} \to A$ the projection.

Recall that $h: A \to A$ is called a twist map if h has a lifting $\bar{h}: \bar{A} \to \bar{A}$ which maps the points of the parallel lines $y = a, y = b$ in opposite directions.

The simplest examples of twist maps are obtained by integrating the flow shown in Figure 1.1.

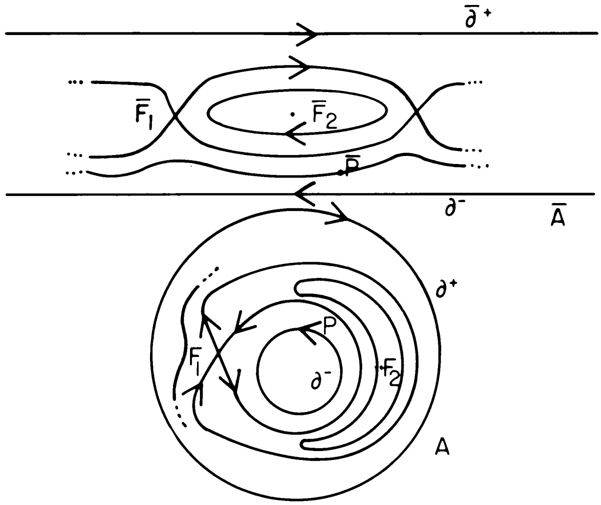

FIGURE 1.1

Notice that h itself can have fixed points on the boundary of A and that the set of twist maps is open in the space of all homeomorphisms of A, furnished with the C^0 topology.

DEFINITION 1.1. Let $h: A \to A$, $\bar{h}: \bar{A} \to \bar{A}$ be a twist map and assume the fixed point set of $\bar{h}$, Fix $\bar{h}$, is not empty and does not separate $\bar{A}$; let $\bar{A}_0 = \bar{A} -$ Fix $\bar{h}$, $\bar{h}_0 = \bar{h}|\bar{A}_0$ and $\tilde{A}_0$ be the universal cover of $\bar{A}_0$. *We say $h: A \to A$ satisfies condition* L if $\bar{h}_0: \bar{A}_0 \to \bar{A}_0$ has a *lifting* $\tilde{h}_0: \tilde{A}_0 \to \tilde{A}_0$ which sends every boundary component of $\partial\tilde{A}_0$ onto itself.

EXAMPLE. It is clear that any diffeomorphism obtained by integrating the flow of Figure 1.1 satisfies condition L.

We remark that condition L is a purely homotopy-theoretical condition: it holds if and only if the maps induced by $\bar{h}$ on $\Pi_1(\bar{A}_0, \bar{\partial}^+)$ and $\Pi_1(\bar{A}_0, \bar{\partial}^-)$ are the identity; here $\bar{\partial}^+, \bar{\partial}^-$ denote the upper and lower boundary components of $\bar{A}_0$ and $\Pi_1(X, Y)$ denotes the homotopy classes of arcs in X with endpoints in Y; the homotopies should keep endpoints in Y.

We now assume without loss of generality that h, $\bar{h}$ is a twist map such that $\bar{h}$ sends points of the lower boundary $\bar{\partial}^-$ of $\bar{A}$ to the left. Let $T: \bar{A} \to \bar{A}$ be the generator of the covering transformation group which sends all points of $\bar{A}$ to the *right*. If $P \in A$ is a periodic point of h of period n and $\bar{P} \in \bar{A}$ lies above P, then there exists a unique integer $m = m(P)$ such that $T^m(\bar{h}^n(\bar{P})) = \bar{P}$ and m does not depend on which point $\bar{P}$ lying over P we chose.

Let $p: \bar{A} \to A$ be the covering projection and $h: A \to A$, $\bar{h}: \bar{A} \to \bar{A}$ a twist map (such that $\bar{h}$ sends points of $\bar{\partial}^-$ to the left).

DEFINITION 1.2. We say the periodic point P of h (of period n) belongs to the set $\mathcal{P}_0$ if in $A_0 = A - p(\text{Fix } \bar{h})$ there exists an arc γ from P to ∂^- such that

(a) $m(P) > 0$,

(b) γ and $h^n(\gamma)$ are homotopic in A_0 by a homotopy, which keeps P fixed and the endpoints of γ and $h^n(\gamma)$ on ∂^-,

(c) γ has a lifting, $\bar{\gamma}$, to $\bar{A}$, such that

$$\bar{h}(\bar{\gamma}) \cap \bar{\gamma} = \varnothing.$$

For example, in Figure 1.1 one can easily find a diffeomorphism such that P is a periodic point $\in \mathcal{P}_0$ with $m(P) = 1$.

If the boundary component ∂^- were a fixed point then (b) means that P and ∂^- lie in the same Nielsen fixed point class with respect to the map $h_0^n: A_0 \to A_0$.

Recall that $h: A \to A$ is called topologically transitive if some point has a dense orbit with respect to h. It is an elementary fact that a nonempty, open, invariant set Ω (i.e. $h(\Omega) \subset \Omega$) of a transitive map is dense in A.

We can now state

THEOREM. *Let $h: A \to A$, $\bar{h}: \bar{A} \to \bar{A}$ be a twist map which satisfies condition* L *and such that $\bar{h}$ sends points of $\bar{\partial}^-$ to the left. Then there exists an open neighborhood Ω of ∂^+ such that $\mathcal{P}_0 \cap \Omega = \varnothing$. In particular, if h is also transitive, then $\mathcal{P}_0$ is nowhere dense in A (i.e., its complement contains an open dense subset of A).*

REMARK 1. Ω can be taken to be the subset of all $x \in A_0 \subset A$ which can be joined to ∂^+ by a free arc α, i.e., an arc α such that $\alpha \cap h(\alpha) = \varnothing$.

REMARK 2. If $\mathcal{P}_0(N) = \{P \in \mathcal{P}_0 | \text{ period of } P \leqslant N\}$ then for each N the theorem is obvious with respect to $\mathcal{P}_0(N)$. *Thus the point is that an Ω independent of N exists.*

REMARK 3. The theorem is a *purely topological theorem*; no differentiability or measure-preserving assumptions of any kind are required for h.

Let h now be at least C^1 and let P be a hyperbolic periodic point (of period n) of h; then the sets

$$W^s = W^s(P) = \left\{x \in A | \lim_{k \to +\infty} h^{nk}(x) = P\right\}$$

and

$$W^u = W^u(P) = \left\{x \in A | \lim_{k \to -\infty} h^{nk}(x) = p\right\}$$

are called, respectively, the stable and unstable manifolds of P. It is well known that W^s and W^u are the images of C^1 injective immersions $R \to A$. A point of intersection of $W^s(P)$ and $W^u(P)$, other than P, is called a *homoclinic* point of P.

Let H be a homoclinic point of P (with respect to $h: A \to A$) and let β^s (β^u) be the arcs, in W^s (W^u), from H to P. Assume $P \in \mathcal{P}_0$ and let γ be as in Definition 1.2.

DEFINITION 1.3. We say H is inessential (we write $H \sim 0$) if the arc $\gamma\beta^u\beta^s\gamma^{-1}$ has a lifting to $\bar{A}_0$ which is homotopic to 0 in $\pi_1(\bar{A}_0, \bar{\partial}^-)$ keeping endpoints on $\bar{\partial}^-$.

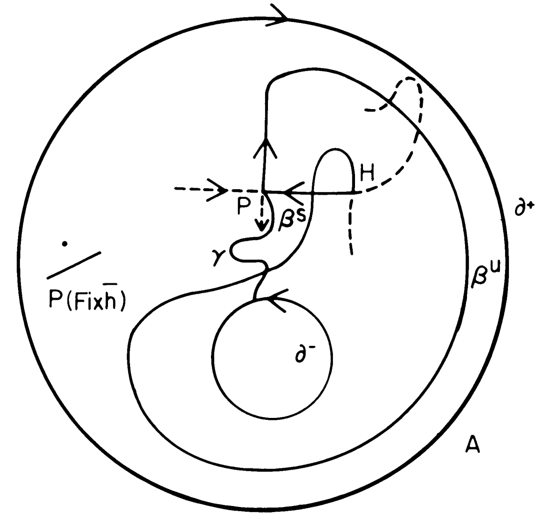

FIGURE 1.2

Define $H_0(P)$ as the set of inessential homoclinic points of P.

COROLLARY TO THE THEOREM. *If h is C^1 and satisfies the hypothesis of the Theorem, then there exists an open neighborhood Ω of ∂^+ such that $\mathcal{H}_0 \cap \Omega = \varnothing$; here $\mathcal{H}_0 = \bigcup \mathcal{H}_0(P)$, the union taken over all hyperbolic $P \in \mathcal{P}_0$. (We can take Ω as in the theorem*; see Remark 1.)

2. Proof of the Theorem. Since Fix $\bar{h} \neq \varnothing$, the boundary of $\tilde{A}_0$, $\partial\tilde{A}_0$, consists of an infinite number of components, each homeomorphic to the real line. We denote a component of $\partial\tilde{A}_0$ lying over ∂^+ or ∂^- by $\tilde{\partial}^+$ or $\tilde{\partial}^-$, and call them opposite boundaries if they lie above opposite boundaries of A. We know there exists a lifting $\tilde{h}_0$: $\tilde{A}_0 \to \tilde{A}_0$ of $\bar{h}_0$: $\bar{A}_0 \to \bar{A}_0$ which is orientation-preserving, fixed point free and takes each boundary component of $\partial\tilde{A}_0$ onto itself; therefore we can draw on each such component $\tilde{\partial}$ an arrow which indicates in which direction $\tilde{h}_0|\tilde{\partial}$ moves the points of $\tilde{\partial}$.

ASSERTION 1. If λ is any arc in $\tilde{A}_0$ joining opposite boundaries $\tilde{\partial}^+$, $\tilde{\partial}^-$ with endpoints $\tilde{Q}^+$, $\tilde{Q}^-$, then $\tilde{h}_0(\tilde{Q}^+)$ and $\tilde{h}_0(\tilde{Q}^-)$ lie in different (unbounded) components of $\tilde{A}_0 - \lambda$; i.e., the arrows are as shown in Figure 2.1.

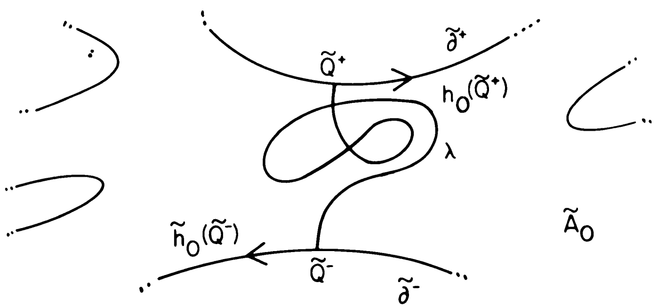

FIGURE 2.1

Indeed, since h is a twist map we can orient $\bar{A}_0$ in such a way that the direction in which $\bar{h}_0$ sends the boundaries $\bar{\partial}^+$, $\bar{\partial}^-$ coincides with the induced orientations on them. Since $\tilde{A}_0$ covers $\bar{A}_0$ and $\tilde{h}_0$ is an orientation-preserving lifting, the same will hold for any two boundaries $\tilde{\partial}^+$, $\tilde{\partial}^-$ lying over $\bar{\partial}^+$, $\bar{\partial}^-$.

Let $P \in \mathcal{P}_0$ via the arc γ and assume without loss of generality[3] that $m(P) = 1$ (see Definition 1.2). If P were sufficiently close to ∂^+, then there would exist an arc $\alpha \subset A_0$ from P to ∂^+, and a lifting, $\bar{\alpha} \subset \bar{A}_0$, to the band $\bar{A}$ of α such that $\bar{h}_0(\bar{\alpha}) \cap \bar{\alpha} = \varnothing$ (Figure 2.2). This is clear because $\bar{h}_0|\bar{\partial}^+$ is equivariant and fixed point free and ∂^+ is compact.

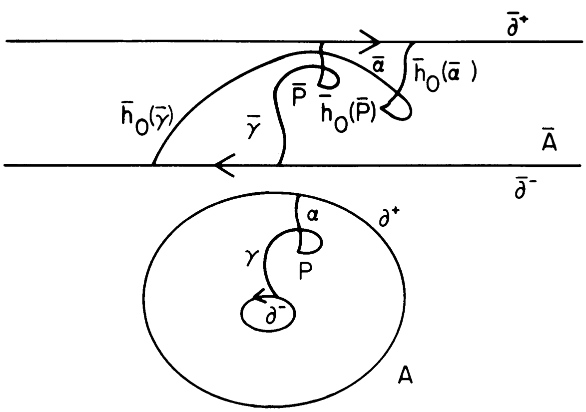

FIGURE 2.2

Assuming the existence of α, we will arrive at a contradiction:

Let Q^- be the initial point of γ; we pick a boundary component $\tilde{\partial}_0$ of $\partial\tilde{A}_0$ and on it a point $\tilde{Q}_0^-$ lying over Q^-. Lift the arc γ to an arc $\tilde{\gamma}_0 \subset \tilde{A}_0$ with initial point $\tilde{Q}_0^- \in \tilde{\partial}_0^-$; then the endpoint of $\tilde{\gamma}_0$, $\tilde{P}_0$, will lie over P (Figure 2.3). Let $\tilde{Q}_1^-$ be the next point (in the direction indicated) lying over Q^- and let $\tilde{P}_1$ be obtained by again lifting γ to $\tilde{A}_0$, but now with initial point $\tilde{Q}_1^-$, to an arc $\tilde{\gamma}_1$ with endpoint $\tilde{P}_1$, which again lies over P.

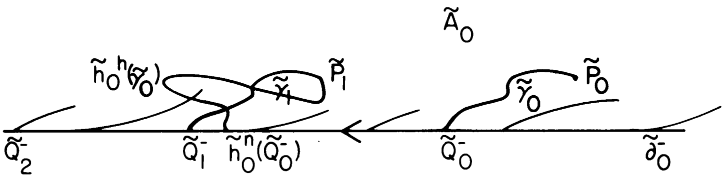

FIGURE 2.3

Similarly obtain $\tilde{Q}_k^-$, $\tilde{\gamma}_k$, $\tilde{P}_k$, also for negative k in the obvious way.

ASSERTION 2. $\tilde{h}_0^{kn}(\tilde{P}_0) = \tilde{P}_k$ for all integers k.

Indeed, by the covering homotopy property, this is an immediate consequence of the facts that $P \in \mathcal{P}_0$ via γ (Definition 1.2(b)) and $\tilde{h}_0$ maps $\tilde{\partial}_0$ onto itself (Figure 2.3).

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[3] In the sense that the proofs for $m(P) > 1$ are entirely similar.

REMARK. The point of Assertion 2 is: With respect to the induced riemannian metric of $\tilde{A}_0$, the iterate $\tilde{h}_0^{kn}(\tilde{P}_0)$ is, for all k, at a *bounded* distance from $\tilde{Q}_k^-$, namely bounded by the length of $\tilde{\gamma}_k$, i.e., the length of γ, which does not depend on k.

Let $\tilde{\partial}_0^+$ be the boundary component of $\partial\tilde{A}_0$ determined by lifting the arc $\gamma \cup \alpha$ (Figure 2.2) to an arc $\lambda = \tilde{\gamma}_0 \cup \tilde{\alpha}_0$ (Figure 2.4) with initial point $\tilde{Q}_0^-$ and endpoint $\tilde{Q}_0^+ \in \tilde{\partial}_0^+$. By Assertion 1, the arrows are as shown in Figure 2.4 and λ divides $\tilde{A}_0$ into two unbounded components.

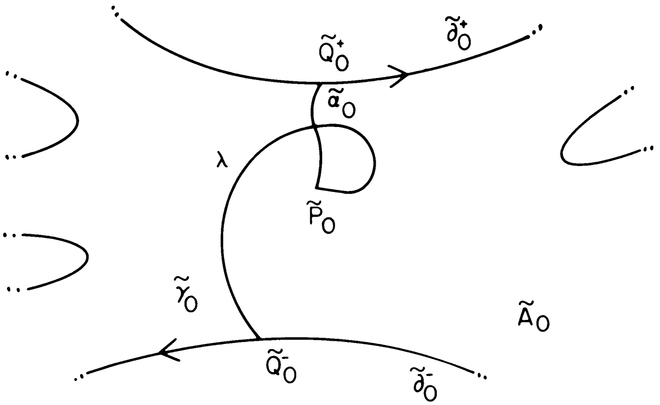

FIGURE 2.4

By Assertion 2 and the compactness of λ and the bounded regions formed by λ, if $k > 0$ is sufficiently large, the points $\tilde{h}_0^{kn}(\tilde{P}_0) = \tilde{P}_k$ and $\tilde{h}_0^{-kn}(\tilde{P}_0) = \tilde{P}_{-k}$ will lie in different, unbounded components of $\tilde{A}_0 - \lambda$, $\tilde{P}_k$ lying in that component where $\tilde{h}_0^{kn}(\tilde{Q}_0^-)$ lies. Moreover, since $\tilde{\alpha}_0$ and $\tilde{\gamma}_0$ are free under $\tilde{h}_0$, by the weak form of Brouwer's Translation Theorem [**1**, Satz 7, p. 13] *they are still free under* $\tilde{h}_0^{kn}$, *where* $k \neq 0$ *is arbitrary*.

We now have the following situation (Figure 2.5).

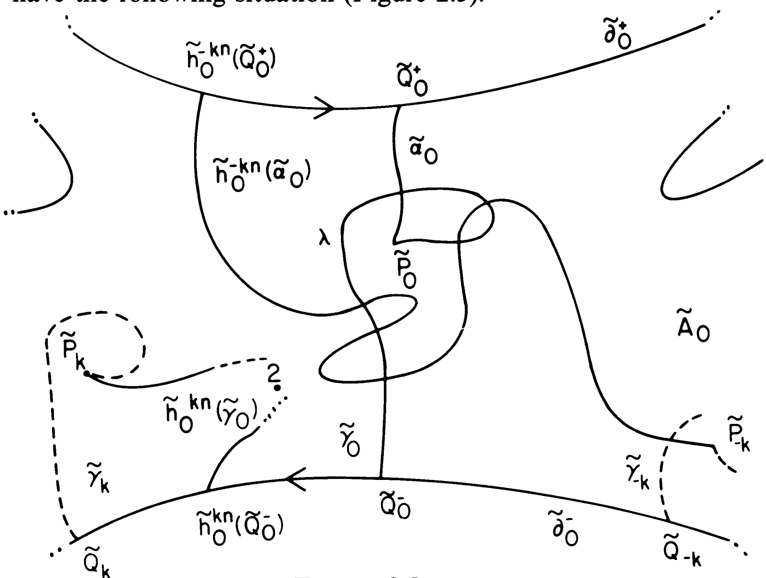

FIGURE 2.5

If $k > 0$ is sufficiently large, the arc $\tilde{h}_0^{-kn}(\tilde{\alpha}_0)$ has to intersect the dividing arc $\lambda = \tilde{\gamma}_0 \cup \tilde{\alpha}_0$ an odd number of times, because by Assertions 1 and 2, its endpoints $\tilde{h}_0^{-kn}(\tilde{Q}_0^+)$ and $\tilde{P}_{-k}$ lie in different unbounded components of $\tilde{A}_0 - \lambda$. Since $\tilde{h}_0^{-kn}(\tilde{\alpha}_0)$ cannot intersect $\tilde{\alpha}_0$ it intersects $\tilde{\gamma}_0$ an odd number of times. Therefore the arc $\tilde{h}_0^{kn}(\tilde{\gamma}_0)$ has to intersect $\tilde{\alpha}_0$ an odd number of times, but since $\tilde{h}_0^{kn}(\tilde{\gamma}_0)$ cannot intersect $\tilde{\gamma}_0$, it intersects the whole dividing arc λ an odd number of times, which is impossible, because its endpoints $\tilde{P}_k$ and $\tilde{h}_0^{kn}(\tilde{Q}_0^-)$ lie in the same component of $\tilde{A}_0 - \lambda$ (Figure 2.5).

We have shown $\mathcal{P}_0$ cannot intersect a small enough neighborhood of ∂^+ and our theorem is proven.

Proof of the corollary. Let $H \in H_0(P)$, and assume an arc α for H exists. By iterating H, if necessary, we can suppose, without loss of generality, that the arc $\gamma \cup \beta^s$ is such that a lifting of it to $\bar{A}_0$ does not intersect its image under $\bar{h}_0$. Let $\tilde{H}_0 \in \tilde{A}_0$ be the endpoint of the lifting, $\tilde{\gamma}_0 \cup \tilde{\beta}_0^s$, of $\gamma \cup \beta_0^s$ to $\tilde{A}_0$ with initial point $\tilde{Q}_0^-$. The facts that $H \sim 0$ (see Definition 1.3) and that β^s, β^u are mapped into themselves under h^n, h^{-n} imply that, for large k, $\tilde{h}_0^{kn}(\tilde{H}_0)$ and $\tilde{h}_0^{-kn}(\tilde{H}_0)$ lie in different unbounded components of $\tilde{A}_0 - \tilde{\gamma}_0 \cup \tilde{\beta}_0^s \cup \tilde{\alpha}_0$ and the argument proceeds as before (Figure 2.6). Here $\tilde{P}_0'$ denotes the endpoint of the lifting, $\tilde{\beta}_0^u$, to $\tilde{A}_0$ of β^u with initial point $\tilde{H}_0$; since $H \sim 0$, $\tilde{P}_0' = \tilde{P}_k$ for some k.

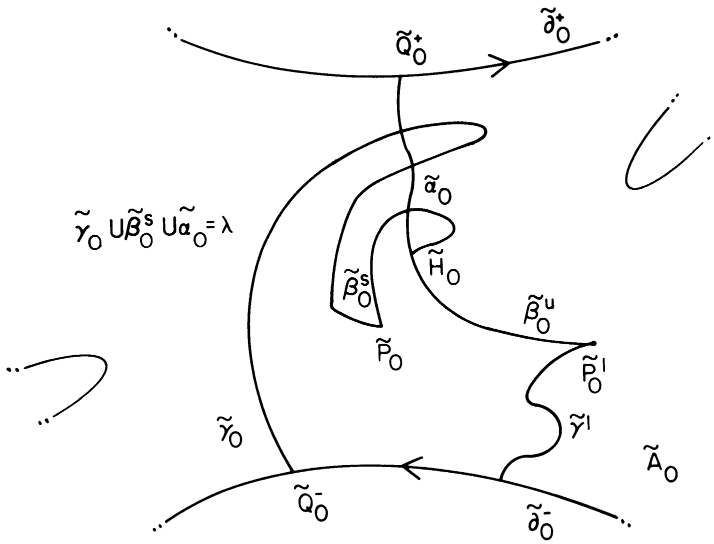

Figure 2.6

Remark. In the proof of the corollary notice that in order to arrive at a contradiction to the existence of α it is enough that the lifting of $\gamma\beta^u\beta^s\gamma^{-1}$ to $\bar{A}_0$ of Definition 1.3 determines a boundary component $\tilde{\partial}_0'$ of $\tilde{A}_0$ which lies in the *right*-hand side (unbounded) component of $\tilde{A}_0 - \tilde{\gamma}_0 \cup \tilde{\beta}_0^s \cup \tilde{\alpha}_0$ (see Figure 2.7).

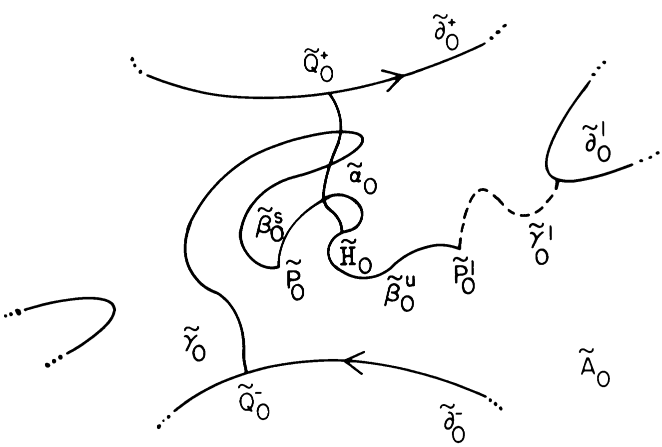

FIGURE 2.7

3. Remarks. (i) In the (not so interesting) case that the twist map h: $A \to A$ is fixed point free, our theorem takes the form:

$$\mathcal{P} \cap S^+ \cap S^- = \varnothing,$$

where $\mathcal{P}$ denotes the set of all periodic points of h and S^+, S^- are the projections into A of the sets $\bar{S}^+$, $\bar{S}^- \subset \bar{A}$ defined as follows: $\bar{X} \in \bar{A}$ lies in $\bar{S}^+$, $\bar{S}^-$ if there exists an arc $\bar{\alpha}^+$ ($\bar{\alpha}^-$) from $\bar{X}$ to $\bar{\partial}^+$ ($\bar{\partial}^-$) such that $\bar{h}(\bar{\alpha}^+) \cap \bar{\alpha}^+ = \varnothing$ ($\bar{h}(\bar{\alpha}^-) \cap \bar{\alpha}^- = \varnothing$).

In the case that h is fixed point free, twist maps seem to have been first related to Brouwer's Translation Theorem by Kerèkjartò, *The plane translation theorem of Brouwer and the last geometric theorem of Poincaré*, Acta Sci. Math. Szeged **4** (1928), 86–102.

(ii) The following is a sufficient *metric* condition for h to satisfy condition L: Assume h: $A \to A$ has only a finite number of fixed points in A and $|h(z) - z| \leqslant \alpha$ for all $z \in A$ for some α such that $0 < \alpha < \Pi a$; then if h does *not* satisfy condition L, there exists a fixed point F such that either:

(a) in an α-neighborhood of F there exists a point z such that the angle between the segments $\overline{zF}$ and $\overline{h(z)F}$ is Π, or

(b) a closed neighborhood of F intersects the boundary of A or a closed 2α-neighborhood of F contains a fixed point different from F.

To see this, it is enough to show that in A_0 there is a unique shortest arc joining any $z \in A_0$ to $h_0(z)$, because then a good $\tilde{h}_0$ will be obtained by lifting h_0 along these arcs. The negation of the conclusions (a), (b) above immediately shows that outside of an α-neighborhood of the inner boundary of A, we can join z to $h_0(z)$, in A_0, by the straight line segment $\overline{zh_0(z)}$. In an α-neighborhood of the inner boundary, since $\alpha < \Pi a$, there also is a unique shortest arc from z to $h_0(z)$ (Figure 3.1).

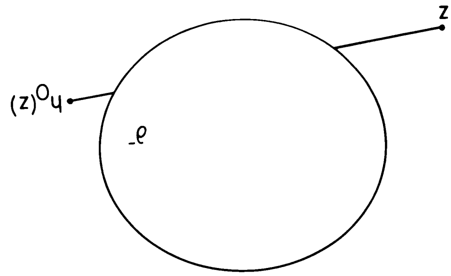

FIGURE 3.1

(iii) Let $h: A \to A$ be a C^1 twist diffeomorphism obtained by integrating the flow of Figure 1.1 such that the fixed points F_1, F_2 are *transversal*, i.e., in $A \times A$ the graph of h intersects the diagonal transversely at F_1 and F_2. We already know h satisfies condition L; however, by making h coincide with a small rotation near the elliptic fixed point F_2 and taking time t small enough, condition L will be satisfied because of the metric condition of (ii) above. It follows that any C^1 $h': A \to A$ sufficiently C^1 close to h will also satisfy condition L, since its corresponding fixed points F_1', F_2', will, due to transversality, be near F_1 and F_2. *Hence, at least in this case, condition* L *is stable under* C^1 *perturbations.*

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